

# SECONDARY CHARACTERISTIC CLASSES FOR SUBGROUPS OF AUTOMORPHISM GROUPS OF FREE GROUPS

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**ABSTRACT.** By analyzing how the Borel regulator classes vanish on various groups related to  $GL(n, \mathbb{Z})$ , we define *three* series of secondary characteristic classes for subgroups of automorphism groups of free groups.

The first case is the IA-automorphism groups and we show that our classes coincide with higher FR torsions due to Igusa. The second case is the mapping class groups and our classes also turn out to be his higher torsions which are non-zero multiples of the Mumford-Morita-Miller classes of *even* indices. Our construction gives new group cocycles for these still mysterious classes. The third case is the outer automorphism groups of free groups of specific ranks. Here we give a conjectural geometric meaning to a series of unstable homology classes called the Morita classes. We expect that certain *unstable* secondary classes would detect them.

## 1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

The Borel regulator classes  $\beta_{2k+1} \in H^{4k+1}(GL(n, \mathbb{Z}); \mathbb{R})$  ( $k = 1, 2, \dots$ ) are stable cohomology classes of  $GL(n, \mathbb{Z})$  and they play fundamental roles in diverse branches of mathematics including number theory, algebraic geometry, differential geometry and topology or more specifically algebraic  $K$ -theory and characteristic classes of flat bundles with arithmetic structure groups.

There are two important groups related to  $GL(n, \mathbb{Z})$ . One is the outer automorphism group of a free group of rank  $n$ , denoted by  $\text{Out } F_n$  and there exists a natural homomorphism from this group onto  $GL(n, \mathbb{Z})$ . The other is the integral symplectic group  $\text{Sp}(2g, \mathbb{Z})$  which can be considered as a natural subgroup of  $GL(2g, \mathbb{Z})$ .

Igusa [28] proved that the pull back of all the Borel classes to  $\text{Out } F_n$  vanish and, more recently, Galatius [19] proved a definitive result that the stable rational cohomology of  $\text{Out } F_n$  is trivial. On the other hand, a classical theorem of Borel [3] which determines the stable rational cohomology groups of both  $GL(n, \mathbb{Z})$  and  $\text{Sp}(2g, \mathbb{Z})$  implies that the restriction of the Borel classes to the latter group vanish.

By making use of these vanishing of the Borel regulator classes  $\beta_{2k+1}$  in various ways, we define three series of secondary characteristic classes for subgroups of the automorphism groups of free groups.

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First, in §3 we use the vanishing of  $\beta_{2k+1}$  on  $\text{Out } F_n$ , due to Igusa and Galatius, to define a secondary class

$$T\beta_{2k+1} \in H^{4k}(\text{IOut}_n; \mathbb{R})$$

(see Definition 3.2), where  $\text{IOut}_n$  denotes the subgroup of  $\text{Out } F_n$  consisting of all the elements which act on the abelianization of  $F_n$  trivially.

Second, in §4 we compare the above vanishing on  $\text{Out } F_n$  with that on  $\text{Sp}(2g, \mathbb{Z})$  for the case  $n = 2g$  to define another secondary class

$$\hat{\beta}_{2k+1} \in H^{4k}(\mathcal{M}_{g,*}; \mathbb{R})$$

(see Definition 4.2), where  $\mathcal{M}_{g,*}$  denotes the mapping class group of a genus  $g$  closed surface with a base point and it can be considered as a subgroup of  $\text{Out } F_{2g}$ .

We show that  $T\beta_{2k+1}$  is nothing other than the higher FR torsion  $\tau_{2k}(\text{IOut}_n)$  of the group  $\text{IOut}_n$  due to Igusa [28] (see Theorem 3.5). We also show that  $\hat{\beta}_{2k+1}$  is the same as his higher torsion class and this implies that it is a non-zero multiple of the MMM class  $e_{2k}$  of even indices (see Theorem 4.7). Our construction gives a direct relation between the Borel classes and the MMM classes of even indices.

Finally in §5, which is the “heart” of the present paper, we give a conjectural geometric meaning of the Morita classes  $\mu_k$ , which make a series of unstable homology classes of  $\text{Out } F_n$ . This is done by considering certain *unstable* refinements of the above construction (see Conjecture 5.7 and Conjecture 5.9). This would give a possible way of proving non-triviality of these classes. At present, only the first three classes are known to be non-trivial ([45][11][21]). We hope that the results in §3 and §4 would serve as supporting evidence for the above conjectures.

The present work began by a trial to prove our conjecture on possible geometric meaning of the Morita classes as sketched in Remark 9.6 of [48] and further discussed in §5 of the present paper. Meanwhile we found that, by using simpler ideas we can define certain secondary characteristic classes for the IA automorphism groups as well as the mapping class groups. Then we noticed that our secondary classes are nothing other than the higher FR torsion classes due to Igusa (and Klein for the Torelli group) developed in [28] as stated above. Our results could have been obtained right after Galatius’ paper [19] appeared. Indeed, it may be said that our construction “realizes” Igusa’s higher FR torsions from the viewpoint of the theory of group cohomology in the cases of the above two kinds of groups, based on the vanishing theorem of Galatius.

We hope that our construction would shed a new light on the difficult open problems of determining whether Igusa’s higher torsion classes as well as the MMM classes of even indices are non-trivial on  $\text{IOut}_n$  and the Torelli group, respectively. These two classes are *stable* characteristic classes in the sense that they are defined for all ranks or genera. On the other hand, the initial problem of giving a geometric meaning to the Morita classes treats the exact boundary line between the stable and unstable ranges for the Borel classes.

## 2. PRELIMINARIES

In this section, we recall basic theorems concerning the stable cohomology of various groups which we use in this paper. They are,  $\mathrm{GL}(n, \mathbb{Z})$ ,  $\mathrm{Sp}(2g, \mathbb{Z})$ ,  $\mathrm{Aut} F_n$  and the mapping class group  $\mathcal{M}_g$ . First, we have the following classical result.

**Theorem 2.1** (Borel [3]). *The cohomology groups  $H^*(\mathrm{GL}(n, \mathbb{Z}); \mathbb{Q})$  and  $H^*(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{Q})$  stabilize, with respect to  $n$  and  $g$  respectively, and the stable cohomology groups are given as follows.*

$$\begin{aligned} \lim_{n \rightarrow \infty} H^*(\mathrm{GL}(n, \mathbb{Z}); \mathbb{R}) &\cong \wedge_{\mathbb{R}}(\beta_3, \beta_5, \dots) \\ \lim_{n \rightarrow \infty} H^*(\mathrm{Sp}(n, \mathbb{Z}); \mathbb{Q}) &\cong \mathbb{Q}[c_1, c_3, \dots]. \end{aligned}$$

Here

$$\beta_{2k+1} \in H^{4k+1}(\mathrm{GL}(n, \mathbb{Z}); \mathbb{R}) \quad (k = 1, 2, \dots)$$

denote the Borel regulator classes and  $c_1, c_3, \dots, c_{2i-1}, \dots \in H^*(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{Q})$  denote the Chern classes of the universal  $g$ -dimensional complex vector bundle over the classifying space of the group  $\mathrm{Sp}(2g, \mathbb{Z})$ . The Borel classes are stable classes in the sense that they are pull backs of the Borel regulator classes  $\beta_k \in H^{2k-1}(\mathrm{BGL}(\infty, \mathbb{C})^\delta; \mathbb{C})$  ( $k = 1, 2, \dots$ ) under the natural inclusion  $i : \mathrm{GL}(n, \mathbb{Z}) \rightarrow \mathrm{GL}(\infty, \mathbb{C})^\delta$  where

$$\mathrm{GL}(\infty, \mathbb{C})^\delta = \lim_{n \rightarrow \infty} \mathrm{GL}(n, \mathbb{C})^\delta$$

equipped with the *discrete* topology. See [15][5] for geometric background of these classes. We recall a property of the Borel classes for later use.

**Proposition 2.2** (see Dupont-Hain-Zucker [16]). *The Borel regulator class  $\beta_{2k+1}$  is primitive. Namely, if we denote by*

$$\bar{\mu} : \mathrm{GL}(n, \mathbb{Z}) \times \mathrm{GL}(n', \mathbb{Z}) \rightarrow \mathrm{GL}(n + n', \mathbb{Z})$$

*the natural inclusion, then we have*

$$\bar{\mu}^*(\beta_{2k+1}) = \beta_{2k+1} \otimes 1 + 1 \otimes \beta_{2k+1}.$$

Next we consider the (outer) automorphism groups of free groups  $\mathrm{Aut} F_n$  and  $\mathrm{Out} F_n$ . Hatcher and Vogtmann [26] (see also [27]) proved that homology groups of these groups stabilize and Galatius determined the stable homology groups as follows. Let  $\mathrm{Aut}_\infty$  denote  $\lim_{n \rightarrow \infty} \mathrm{Aut} F_n$  and let  $QS^0 = \lim_{n \rightarrow \infty} \Omega^n S^n$ .

**Theorem 2.3** (Galatius [19]). *There exists a homology equivalence (H.e. for short)*

$$\mathbb{Z} \times \mathrm{BAut}_\infty \xrightarrow{\text{H.e.}} QS^0.$$

*In particular*

$$\lim_{n \rightarrow \infty} H^*(\mathrm{Aut} F_n; \mathbb{Q}) \cong \lim_{n \rightarrow \infty} H^*(\mathrm{Out} F_n; \mathbb{Q}) \cong \mathbb{Q}.$$

Finally we consider the mapping class groups. Let  $\mathcal{M}_g$  denote the mapping class group of a closed oriented surface  $\Sigma_g$  of genus  $g$ . Also let  $\mathcal{M}_{g,*}$  and  $\mathcal{M}_{g,1}$  be the mapping class groups of  $\Sigma_g$  relative to a base point and embedded disk, respectively. Harer [23] proved that the (co)homology groups of the mapping class groups stabilize and Madsen and Weiss determined the stable rational cohomology as follows.

**Theorem 2.4** (Madsen-Weiss [40]).

$$\lim_{g \rightarrow \infty} H^*(\mathcal{M}_g; \mathbb{Q}) \cong \lim_{g \rightarrow \infty} H^*(\mathcal{M}_{g,1}; \mathbb{Q}) \cong \mathbb{Q}[\text{MMM classes}].$$

We also have the following result.

**Theorem 2.5** (see Harer [24], Looijenga [39] and Madsen-Weiss [40]).

$$\lim_{g \rightarrow \infty} H^*(\mathcal{M}_{g,*}; \mathbb{Q}) \cong \mathbb{Q}[e, \text{MMM classes}].$$

As for the stable ranges, we refer to [3][32][55] for  $\text{GL}(n, \mathbb{Z})$ , [3][4] for  $\text{Sp}(2g, \mathbb{Z})$ , [26][27] for the (outer) automorphism groups of free groups and [56] as well as references therein for the mapping class groups.

### 3. SECONDARY CHARACTERISTIC CLASSES FOR THE IA AUTOMORPHISM GROUP

In this section, we consider the IA automorphism group, denoted by  $\text{IA}_n$  which is defined to be the kernel of the natural projection  $p : \text{Aut } F_n \rightarrow \text{GL}(n, \mathbb{Z})$ . Thus we have the following exact sequence

$$1 \rightarrow \text{IA}_n \xrightarrow{i} \text{Aut } F_n \xrightarrow{p} \text{GL}(n, \mathbb{Z}) \rightarrow 1.$$

The outer automorphism group of  $F_n$  is defined as  $\text{Out } F_n = \text{Aut } F_n / \text{Inn } F_n$  and we have the following similar exact sequence

$$1 \rightarrow \text{IOut}_n \xrightarrow{i} \text{Out } F_n \xrightarrow{p} \text{GL}(n, \mathbb{Z}) \rightarrow 1.$$

The first of our three secondary characteristic classes are elements of  $H^*(\text{IA}_n; \mathbb{R})$  and  $H^*(\text{IOut}_n; \mathbb{R})$  defined as follows. Let

$$b_{2k+1} \in Z^{4k+1}(\text{GL}(n, \mathbb{Z}); \mathbb{R})$$

be a  $(4k+1)$ -cocycle of the group  $\text{GL}(n, \mathbb{Z})$  which represents the Borel regulator class  $\beta_{2k+1} \in H^{4k+1}(\text{GL}(n, \mathbb{Z}); \mathbb{R})$ . We know by Igusa that

$$p^* \beta_{2k+1} = 0 \in H^{4k+1}(\text{Out } F_n; \mathbb{R}).$$

Hence we can choose a  $4k$ -cochain

$$z_{4k} \in C^{4k}(\text{Out } F_n; \mathbb{R})$$

such that  $\delta z_{4k} = p^* \beta_{2k+1}$ . The restriction  $i^* z_{4k}$  of  $z_{4k}$  to the subgroup  $\text{IOut}_n \subset \text{Out } F_n$  is a cocycle and we can consider its cohomology class

$$[i^* z_{4k}] \in H^{4k}(\text{IOut}_n; \mathbb{R}).$$

**Proposition 3.1.** *The cohomology class  $[i^* z_{4k}] \in H^{4k}(\text{IOut}_n; \mathbb{R})$  is well-defined independent of the choices of  $b_{2k+1}$  and  $z_{4k}$  in the stable range  $n \geq 2k + 4$  of  $\text{Out } F_n$ . Furthermore it is  $\text{GL}(n, \mathbb{Z})$ -invariant so that*

$$[i^* z_{4k}] \in H^{4k}(\text{IOut}_n; \mathbb{R})^{\text{GL}(n, \mathbb{Z})}.$$

*Proof.* First we prove the former part of the claim. Let  $b'_{2k+1} \in Z^{4k+1}(\text{GL}(n, \mathbb{Z}); \mathbb{R})$  be another representative of the Borel class  $\beta_{2k+1}$  and let

$$z'_{4k} \in C^{4k}(\text{Out } F_n; \mathbb{R})$$

be a cochain such that  $\delta z'_{4k} = p^* b'_{2k+1}$ . Now there exists an element  $u \in C^{4k}(\text{GL}(n, \mathbb{Z}); \mathbb{R})$  such that

$$\delta u = b'_{2k+1} - b_{2k+1}.$$

Then we have

$$\delta z'_{4k} = p^* b'_{2k+1} = p^*(b_{2k+1} + \delta u) = \delta(z_{4k} + p^* u).$$

It follows that

$$\delta(z'_{4k} - z_{4k} - p^* u) = 0.$$

By the vanishing theorem of Galatius, there exists an element  $v \in C^{4k-1}(\text{Out } F_n; \mathbb{R})$  such that

$$z'_{4k} - z_{4k} - p^* u = \delta v.$$

Then we have

$$i^* z'_{4k} = i^* z_{4k} + \delta i^* v$$

because  $i^* p^* u = 0$ . Hence

$$[i^* z'_{4k}] = [i^* z_{4k}] \in H^{4k}(\text{IOut}_n; \mathbb{R})$$

as required.

Next we prove the latter part claiming that this cohomology class is  $\text{GL}(n, \mathbb{Z})$ -invariant. For this, it is enough to prove the following. Any element  $\varphi \in \text{Out } F_n$  induces an automorphism  $\iota_\varphi$  of  $\text{IOut}_n$  by the correspondence

$$\text{IOut}_n \ni \psi \mapsto \iota_\varphi(\psi) = \varphi \psi \varphi^{-1} \in \text{IOut}_n.$$

Then, under the induced automorphism

$$\iota_\varphi^* : H^*(\text{IOut}_n; \mathbb{R}) \cong H^*(\text{IOut}_n; \mathbb{R}),$$

the equality

$$\iota_\varphi^*([i^* z_{4k}]) = [i^* z_{4k}]$$

holds. To prove this, observe first that the cohomology class  $\iota_\varphi^*([i^* z_{4k}])$  is represented by the cocycle  $i^* \iota_\varphi^* z_{4k}$  which is the restriction to  $\text{IOut}_n$  of the cochain

$$\iota_\varphi^* z_{4k} \in Z^{4k}(\text{Out } F_n; \mathbb{R}).$$

This cochain in turn satisfies the identity

$$\delta(\iota_\varphi^* z_{4k}) = \iota_\varphi^* p^* b_{2k+1}.$$

If we denote by  $\iota_{\bar{\varphi}}$  the inner automorphism of  $\mathrm{GL}(n, \mathbb{Z})$  induced by the projected element  $\bar{\varphi} = p(\varphi) \in \mathrm{GL}(n, \mathbb{Z})$ , then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Out} F_n & \xrightarrow{\iota_{\varphi}} & \mathrm{Out} F_n \\ p \downarrow & & p \downarrow \\ \mathrm{GL}(n, \mathbb{Z}) & \xrightarrow{\iota_{\bar{\varphi}}} & \mathrm{GL}(n, \mathbb{Z}). \end{array}$$

Then we have  $\iota_{\varphi}^* p^* b_{2k+1} = p^* \iota_{\bar{\varphi}}^* b_{2k+1}$  and hence

$$\delta(\iota_{\varphi}^* z_{4k}) = p^* \iota_{\bar{\varphi}}^* b_{2k+1}.$$

Now, as is well known, any inner automorphism of any group induces the identity on its (co)homology group. Therefore, the cochain  $\iota_{\bar{\varphi}}^* b_{2k+1}$  of the group  $\mathrm{GL}(n, \mathbb{Z})$  is cohomologous to  $b_{2k+1}$ . Hence, by replacing  $b'_{2k+1}$  and  $z'_{4k}$  with  $\iota_{\bar{\varphi}}^* b_{2k+1}$  and  $\iota_{\bar{\varphi}}^* z_{4k}$  respectively, in the former argument above, we can conclude that

$$\iota_{\varphi}^*([i^* z_{4k}]) = [i^* \iota_{\varphi}^* z_{4k}] = [i^* z_{4k}]$$

as required. This completes the proof.  $\square$

**Definition 3.2.**

$$T\beta_{2k+1} = [i^* z_{4k}] \in H^{4k}(\mathrm{IOut}_n; \mathbb{R})^{\mathrm{GL}(n, \mathbb{Z})}$$

$$T\beta_{2k+1}^0 = q^*[i^* z_{4k}] \in H^{4k}(\mathrm{IA}_n; \mathbb{R})^{\mathrm{GL}(n, \mathbb{Z})}$$

where

$$q : \mathrm{IA}_n \rightarrow \mathrm{IOut}_n$$

denotes the natural projection. By the above construction, we see that our secondary class  $T\beta_{2k+1}^0$  is stable in the following sense. Namely, if we denote by

$$i : \mathrm{IA}_n \rightarrow \mathrm{IA}_{n+1}$$

the natural inclusion in the stable range, then we have

$$i^* T\beta_{2k+1}^0 = T\beta_{2k+1}^0.$$

It follows that we can define this class for *all*  $n$  by just pulling back the above stable class by the natural inclusion  $\mathrm{IA}_n \subset \mathrm{IA}_N$  where  $N$  is a large number.

**Proposition 3.3.** *The class  $T\beta_{2k+1}^0$  is primitive in the following sense. Namely, if we denote by*

$$\mu_0 : \mathrm{IA}_n \times \mathrm{IA}_{n'} \rightarrow \mathrm{IA}_{n+n'}$$

*the natural homomorphism, then we have*

$$\mu_0^*(T\beta_{2k+1}^0) = T\beta_{2k+1}^0 \otimes 1 + 1 \otimes T\beta_{2k+1}^0.$$

*Proof.* Consider the following commutative diagram

$$(1) \quad \begin{array}{ccc} \mathrm{IA}_n \times \mathrm{IA}_{n'} & \xrightarrow{\mu_0} & \mathrm{IA}_{n+n'} \\ i \times i \downarrow & & i \downarrow \\ \mathrm{Aut} F_n \times \mathrm{Aut} F_{n'} & \xrightarrow{\mu} & \mathrm{Aut} F_{n+n'} \\ p \times p \downarrow & & p \downarrow \\ \mathrm{GL}(n, \mathbb{Z}) \times \mathrm{GL}(n', \mathbb{Z}) & \xrightarrow{\bar{\mu}} & \mathrm{GL}(n+n', \mathbb{Z}). \end{array}$$

By Proposition 2.2, we have

$$\bar{\mu}^* \beta_{2k+1} = \beta_{2k+1} \otimes 1 + 1 \otimes \beta_{2k+1}.$$

It follows that there exists a cochain  $d \in C^{4k-1}(\mathrm{GL}(n, \mathbb{Z}) \times \mathrm{GL}(n', \mathbb{Z}); \mathbb{R})$  such that

$$\bar{\mu}^* b_{2k+1} = b_{2k+1} \times 1 + 1 \times b_{2k+1} + \delta d.$$

Then we have

$$\begin{aligned} (p \times p)^* \bar{\mu}^* b_{2k+1} &= \delta z_{4k} \times 1 + 1 \times \delta z_{4k} + (p \times p)^* \delta d \\ &= \delta (z_{4k} \times 1 + 1 \times z_{4k} + (p \times p)^* d). \end{aligned}$$

Now, by the definition of  $T\beta_{2k+1}$

$$p^* b_{2k+1} = \delta z_{4k} \text{ and } T\beta_{2k+1} = [i^* z_{4k}]$$

so that

$$\mu^* p^* b_{2k+1} = \delta \mu^* z_{4k}.$$

Since  $p \circ \mu = \bar{\mu} \circ p \times p$ , we can conclude

$$\delta (z_{4k} \times 1 + 1 \times z_{4k} + (p \times p)^* d) = \delta \mu^* z_{4k}$$

and hence

$$\delta (\mu^* z_{4k} - z_{4k} \times 1 - 1 \times z_{4k} - (p \times p)^* d) = 0.$$

Thus, the element in the parenthesis above is a cocycle of the group  $\mathrm{Aut} F_n \times \mathrm{Aut} F_{n'}$ . In the stable range, where  $n$  and  $n'$  are sufficiently large

$$H^{4k}(\mathrm{Aut} F_n \times \mathrm{Aut} F_{n'}; \mathbb{R}) = 0$$

by the vanishing theorem of Galatius. Therefore, there exists an element

$$d' \in C^{4k-1}(\mathrm{Aut} F_n \times \mathrm{Aut} F_{n'}; \mathbb{R})$$

such that

$$\mu^* z_{4k} - z_{4k} \times 1 - 1 \times z_{4k} - (p \times p)^* d = \delta d'.$$

Hence

$$\mu_0^* i^* z_{4k} = (i \times i)^* \mu^* z_{4k} = i^* z_{4k} \times 1 + 1 \times i^* z_{4k} + \delta (i \times i)^* d'.$$

We now conclude that

$$\mu_0^* [i^* z_{4k}] = [i^* z_{4k}] \otimes 1 + 1 \otimes [i^* z_{4k}].$$

This completes the proof.  $\square$

**Remark 3.4.** The above proposition is a corollary of the following theorem and general property of the higher FR torsion (see Theorem 5.7.5 of Igusa [28]). However, for completeness, we gave a proof in the framework of this paper. In the next section, it will be extended to the case of mapping class group (Proposition 4.6). See also Remark 4.8.

**Theorem 3.5.** *The secondary class  $T\beta_{2k+1}$  is equal to Igusa's higher FR torsion class  $\tau_{2k}(\text{IOut}_n)$ .*

*Proof.* Proof is given by putting the vanishing theorem of Galatius in Igusa's theory of higher FR torsions developed in [28]. More precisely, let us consider the following homotopy commutative diagram

$$\begin{array}{ccc}
 & & \Omega\text{BGL}(\infty, \mathbb{Z})^+ \\
 & & \downarrow \bar{i}_0 \\
 \text{BOut}_n & \xrightarrow{f_0} & |\mathcal{W}h^h(\mathbb{Z}, 1)| \\
 \downarrow \text{Bi} & & \downarrow \bar{i} \\
 \text{BOut } F_n & \xrightarrow{f} & \mathbb{Z} \times \text{BOut}_\infty^+ \stackrel{h.e.}{\cong} QS^0 \\
 \downarrow \text{Bp} & & \downarrow \bar{p} \\
 \text{BGL}(n, \mathbb{Z}) & \xrightarrow{\bar{f}} & \mathbb{Z} \times \text{BGL}(\infty, \mathbb{Z})^+
 \end{array}$$

described in Proposition 8.5.6 of the above cited book. The point here is that we can put Galatius' result

$$\mathbb{Z} \times \text{BOut}_\infty^+ \stackrel{h.e.}{\cong} QS^0$$

(see Theorem 2.3) in the third place from the top of the right column. Each of the two successive three spaces appearing in the right column is a fibration sequence and each connected component of  $QS^0$  is rationally trivial. Hence the map  $\bar{i}_0$  on the right column is a rational homotopy equivalence. Now Igusa's higher torsion is defined roughly as follows. He constructs an explicit  $4k$ -cocycle of the Volodin space  $V(\mathbb{Z})$ , which is homotopy equivalent to  $\Omega\text{BGL}(\infty, \mathbb{Z})^+$ , such that its cohomology class in  $H^{4k}(\Omega\text{BGL}(\infty, \mathbb{Z})^+; \mathbb{R})$  corresponds to the Borel class  $\beta_{2k+1}$ . Since  $\bar{i}_0$  is a rational homotopy equivalence as above, this cohomology class induces the *universal* higher FR torsion class

$$\tau_{2k} \in H^{4k}(|\mathcal{W}h^h(\mathbb{Z}, 1)|; \mathbb{R}).$$

Then his torsion  $\tau_{2k}(\text{IOut}_n) \in H^{4k}(\text{IOut}_n; \mathbb{R})$  is defined to be the image under  $f_0^*$  of the above universal class. Our claim now follows by simply comparing the spectral sequence for the rational cohomology groups of the fibration given by the lower three terms of the right column with that of the path fibration over  $\text{BGL}(\infty, \mathbb{Z})^+$ .  $\square$

**Problem 3.6.** Construct a cochain  $z_{4k} \in C^{4k}(\text{Out } F_n; \mathbb{R})$  such that  $\delta z_{4k} = p^* b_{2k+1}^H$  explicitly. Here  $b_{2k+1}^H$  denotes Hamida's cocycle given in [22] which represents the Borel class  $\beta_{2k+1}$ . We can also consider another cocycle for  $\beta_{2k+1}$  along the line of Dupont [15].



## 4. SECONDARY CHARACTERISTIC CLASSES FOR THE MAPPING CLASS GROUP

In this section, we define secondary classes for the mapping class group by comparing two different ways of vanishing of the Borel regulator classes  $\beta_{2k+1}$ , one on the automorphism group of free groups and the other on the Siegel modular group. We show that they are non-zero multiples of the MMM classes  $e_{2k}$  of *even* indices by relating them to Igusa's higher torsions for the mapping class groups. This would give a new geometric meaning to these classes from the viewpoint of the theory of cohomology of groups.

We mention here that, about differences between  $e_{\text{odd}}$  and  $e_{\text{even}}$  classes, there are several interesting results. Church, Farb and Thibault [8] proved that the former classes have some nice geometric property which the latter classes do not. On the other hand, Giansiracusa and Tillmann [20] and Sakasai [53] proved that the former classes vanish on the handlebody subgroup and Lagrangian subgroup of the mapping class group, respectively. More strongly, Hatcher [25] proved that the stable rational cohomology of the handlebody subgroup is the polynomial algebra generated by  $e_{\text{even}}$  classes. Also it was shown in [45] that these classes represent the *orbifold* Pontrjagin classes of the moduli space of curves.

Now let  $\mathcal{M}_{g,1}$  denote the mapping class group of a compact oriented genus  $g$  surface with one boundary component as before and let

$$i : \mathcal{M}_{g,1} \rightarrow \text{Aut } F_{2g}$$

be the inclusion given by the classical theorem of Dehn-Nielsen-Zieschang. It induces a related inclusion

$$i : \mathcal{M}_{g,*} \rightarrow \text{Out } F_{2g}$$

and we have the following commutative diagrams.

$$\begin{array}{ccccc} \mathcal{I}_{g,*} & \xrightarrow{j_0} & \text{IOut}_{2g} & & \mathcal{M}_{g,1} & \xrightarrow{\tilde{j}} & \text{Aut } F_{2g} \\ i_0 \downarrow & & i \downarrow & & q \downarrow & & \bar{q} \downarrow \\ \mathcal{M}_{g,*} & \xrightarrow{j} & \text{Out } F_{2g} & & \mathcal{M}_{g,*} & \xrightarrow{j} & \text{Out } F_{2g} \\ p_0 \downarrow & & i \downarrow & & p_0 \downarrow & & p \downarrow \\ \text{Sp}(2g, \mathbb{Z}) & \xrightarrow{\tilde{j}} & \text{GL}(2g, \mathbb{Z}) & & \text{Sp}(2g, \mathbb{Z}) & \xrightarrow{\tilde{j}} & \text{GL}(2g, \mathbb{Z}). \end{array}$$

As in the previous section, there exists a cochain  $z_{4k} \in C^{4k}(\text{Out } F_{2g}; \mathbb{R})$  such that

$$p^*b_{2k+1} = \delta z_{4k}.$$

On the other hand, the stable cohomology of  $\text{Sp}(2g, \mathbb{Z})$  is a polynomial algebra on  $c_1, c_3, \dots$  (see Theorem 2.1) so that there are no cohomology classes of odd degrees. Hence there exists a cochain  $y_{4k} \in C^{4k}(\text{Sp}(2g, \mathbb{Z}); \mathbb{R})$  such that

$$\bar{j}^*(b_{2k+1}) = \delta y_{4k}.$$

Then we have

$$0 = j^*(p^*b_{2k+1}) - p_0^*(\bar{j}^*b_{2k+1}) = \delta(j^*z_{4k} - p_0^*y_{4k})$$

so that

$$j^* z_{4k} - p_0^* y_{4k} \in Z^{4k}(\mathcal{M}_{g,*}; \mathbb{R}).$$

In the stable range, the cohomology class represented by this cocycle can be written as

$$[j^* z_{4k} - p_0^* y_{4k}] = f(e_1, \dots, e_{2k}) + eg(e, e_1, \dots, e_{2k-1}) \in H^{4k}(\mathcal{M}_{g,*}; \mathbb{R})$$

by Theorem 2.5. Also, we have

$$q^*[j^* z_{4k} - p_0^* y_{4k}] = f(e_1, \dots, e_{2k}) \in H^{4k}(\mathcal{M}_{g,1}; \mathbb{R})$$

where

$$q : \mathcal{M}_{g,1} \rightarrow \mathcal{M}_{g,*}$$

denotes the natural projection.

Now it was proved in [42][50] that the homomorphism

$$p_0^* : H^*(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{Q}) \rightarrow H^*(\mathcal{M}_{g,*}; \mathbb{Q})$$

is injective, in a certain stable range, and its image is equal to the subalgebra generated by the MMM classes of *odd* indices. In fact, the pull back of the Chern classes are those of the Hodge bundle over the moduli space and the totality of them is the same as that of the MMM classes of *odd* indices. Hence, by adding suitable cocycles in  $Z^{4k}(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{R})$ , we may assume that the polynomial  $f(e_1, \dots, e_{2k})$  does not contain monomials of  $e_{\text{odd}}$  classes.

**Proposition 4.1.** *The cohomology class  $[j^* z_{4k} - p_0^* y_{4k}] \in H^{4k}(\mathcal{M}_{g,*}; \mathbb{R})$  is well-defined independent of the choices of  $b_{2k+1}$ ,  $z_{4k}$  and  $y_{4k}$ .*

*Proof.* Let  $b'_{2k+1} \in Z^{4k+1}(\mathrm{GL}(2g, \mathbb{Z}); \mathbb{R})$  be another representative of the Borel class  $\beta_{2k+1}$  and let

$$z'_{4k} \in C^{4k}(\mathrm{Out} F_{2g}; \mathbb{R}), \quad y'_{4k} \in C^{4k}(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{R})$$

be cochains such that  $\delta z'_{4k} = p^* b'_{2k+1}$  and  $\delta y'_{4k} = \bar{j}^* b'_{2k+1}$ . We further assume that, in the expression of the cohomology class

$$[j^* z'_{4k} - p_0^* y'_{4k}] = f'(e_1, \dots, e_{2k}) + eg'(e, e_1, \dots, e_{2k-1}) \in H^{4k}(\mathcal{M}_{g,*}; \mathbb{R}),$$

the polynomial  $f'(e_1, \dots, e_{2k})$  does not contain any monomial of  $e_{\text{odd}}$  classes.

Then as in the proof of Proposition 3.1, there exist elements  $u \in C^{4k}(\mathrm{GL}(2g, \mathbb{Z}); \mathbb{R})$  and  $v \in C^{4k-1}(\mathrm{Out} F_{2g}; \mathbb{R})$  such that

$$b'_{2k+1} = b_{2k+1} + \delta u, \quad z'_{4k} = z_{4k} + p^* u + \delta v.$$

Hence

$$\delta y'_{4k} = \bar{j}^* b'_{2k+1} = \bar{j}^* (b_{2k+1} + \delta u) = \delta(y_{4k} + \bar{j}^* u).$$

Therefore, if we set

$$w = y'_{4k} - y_{4k} - \bar{j}^* u,$$

then  $w$  is a cocycle of the group  $\mathrm{Sp}(2g, \mathbb{Z})$  so that we can consider its cohomology class

$$[w] \in H^{4k}(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{R}).$$

Also

$$y'_{4k} = y_{4k} + \bar{j}^* u + w.$$

Now we have

$$\begin{aligned} j^* z'_{4k} - p_0^* y'_{4k} &= j^*(z_{4k} + p^* u + \delta v) - p_0^*(y_{4k} + \bar{j}^* u + w) \\ &= (j^* z_{4k} - p_0^* y_{4k}) + \delta j^* v - p_0^* w \end{aligned}$$

because  $j^* p^* u - p_0^* \bar{j}^* u = 0$ . It follows that

$$[j^* z'_{4k} - p_0^* y'_{4k}] = [j^* z_{4k} - p_0^* y_{4k}] - p_0^*[w] \in H^{4k}(\mathcal{M}_{g,*}; \mathbb{R}).$$

By the definition of our secondary class  $\hat{\beta}_{2k+1}$ , both the cohomology classes  $[j^* z'_{4k} - p_0^* y'_{4k}]$  and  $[j^* z_{4k} - p_0^* y_{4k}]$  do not contain any monomial of  $e_{\text{odd}}$  classes. On the other hand, as mentioned already above, the cohomology class  $p_0^*[w]$  is a linear combination of such monomials. Hence we conclude that  $p_0^*[w] = 0$  and so

$$[j^* z'_{4k} - p_0^* y'_{4k}] = [j^* z_{4k} - p_0^* y_{4k}] \in H^{4k}(\mathcal{M}_{g,*}; \mathbb{R})$$

as required. This completes the proof.  $\square$

Based on the above discussion, we make the following definition.

**Definition 4.2.**

$$\hat{\beta}_{2k+1} = [j^* z_{4k} - p_0^* y_{4k}] \in H^{4k}(\mathcal{M}_{g,*}; \mathbb{R}),$$

$$\hat{\beta}_{2k+1}^0 = q^*(\hat{\beta}_{2k+1}) \in H^{4k}(\mathcal{M}_{g,1}; \mathbb{R}).$$

By the above construction, we see that our secondary class  $\hat{\beta}_{2k+1}^0$  is stable in the following sense. Namely, if we denote by

$$i : \mathcal{M}_{g,1} \rightarrow \mathcal{M}_{g+1,1}$$

the natural inclusion in the stable range, then we have

$$i^* \hat{\beta}_{2k+1}^0 = \hat{\beta}_{2k+1}^0.$$

It follows that we can define this class for *all*  $g$  by just pulling back the above stable class by the natural inclusion  $\mathcal{M}_{g,1} \subset \mathcal{M}_{G,1}$  where  $G$  is a large number.

**Problem 4.3.** Construct a cochain  $y_{4k} \in C^{4k}(\text{Sp}(2g, \mathbb{Z}); \mathbb{R})$  such that  $\delta y_{4k} = j^* b_{2k+1}^H$  explicitly, where  $b_{2k+1}^H$  denotes Hamida's cocycle as before.

**Remark 4.4.** Let us define subgroups  $\text{Aut}^{\text{sp}} F_{2g} \subset \text{Aut } F_{2g}$  and  $\text{Out}^{\text{sp}} F_{2g} \subset \text{Out } F_{2g}$  by setting

$$\begin{aligned} \text{Aut}^{\text{sp}} F_{2g} &= \{\varphi \in \text{Aut } F_{2g}; \varphi_* \in \text{Sp}(2g, \mathbb{Z})\}, \\ \text{Out}^{\text{sp}} F_{2g} &= \{\varphi \in \text{Out } F_{2g}; \varphi_* \in \text{Sp}(2g, \mathbb{Z})\}. \end{aligned}$$

These subgroups were already defined by Igusa in [28] and they are strictly larger than  $\mathcal{M}_{g,1}$  and  $\mathcal{M}_{g,*}$ . In fact, we have the following exact sequences

$$\begin{aligned} 1 &\rightarrow \text{IA}_{2g} \rightarrow \text{Aut}^{\text{sp}} F_{2g} \rightarrow \text{Sp}(2g, \mathbb{Z}) \rightarrow 1, \\ 1 &\rightarrow \text{IOut}_{2g} \rightarrow \text{Out}^{\text{sp}} F_{2g} \rightarrow \text{Sp}(2g, \mathbb{Z}) \rightarrow 1. \end{aligned}$$

Then our secondary characteristic classes are defined as cohomology classes of these groups so that we can write

$$\begin{aligned}\hat{\beta}_{2k+1}^0 &\in H^{4k}(\mathrm{Aut}^{\mathrm{sp}} F_{2g}; \mathbb{R}), \\ \hat{\beta}_{2k+1} &\in H^{4k}(\mathrm{Out}^{\mathrm{sp}} F_{2g}; \mathbb{R}).\end{aligned}$$

There is a close relationship between our secondary classes  $T\beta_{2k+1}$ ,  $T\beta_{2k+1}^0$  and  $\hat{\beta}_{2k+1}$ ,  $\hat{\beta}_{2k+1}^0$ . More precisely, we have the following result. Let  $\mathcal{I}_{g,*} \subset \mathcal{M}_{g,*}$  and  $\mathcal{I}_{g,1} \subset \mathcal{M}_{g,1}$  be the Torelli subgroups of the mapping class groups. Then we have the following commutative diagrams

$$\begin{array}{ccc} \mathcal{I}_{g,*} & \xrightarrow{j_0} & \mathrm{IOut}_{2g} \\ i_0 \downarrow & & i \downarrow \\ \mathcal{M}_{g,*} & \xrightarrow{j} & \mathrm{Out} F_{2g}, \end{array} \quad \begin{array}{ccc} \mathcal{I}_{g,1} & \xrightarrow{j_0} & \mathrm{IA}_{2g} \\ i_0 \downarrow & & i \downarrow \\ \mathcal{M}_{g,1} & \xrightarrow{j} & \mathrm{Aut} F_{2g}. \end{array}$$

**Proposition 4.5.** *We have the following identities.*

$$\begin{aligned}j_0^*(T\beta_{2k+1}) &= i_0^*(\hat{\beta}_{2k+1}), \\ j_0^*(T\beta_{2k+1}^0) &= i_0^*(\hat{\beta}_{2k+1}^0).\end{aligned}$$

*Proof.* It is enough to prove the first identity because the second one follows from it. By the definition, we have

$$\hat{\beta}_{2k+1} = [j^* z_{4k} - p_0^* y_{4k}].$$

Hence

$$i_0^*(\hat{\beta}_{2k+1}) = [i_0^* j^* z_{4k}]$$

because  $i_0^* p_0^* = 0$ . On the other hand

$$T\beta_{2k+1} = [i^* z_{4k}]$$

so that

$$j_0^*(T\beta_{2k+1}) = [j_0^* i^* z_{4k}] = [i_0^* j^* z_{4k}] = i_0^*(\hat{\beta}_{2k+1})$$

completing the proof. □

**Proposition 4.6.** *The class  $\hat{\beta}_{2k+1}^0$  is primitive in the sense that the equality*

$$\mu_0^*(\hat{\beta}_{2k+1}^0) = \hat{\beta}_{2k+1}^0 \otimes 1 + 1 \otimes \hat{\beta}_{2k+1}^0$$

*holds, where  $\mu_0$  denotes the following natural mapping*

$$\mu_0 : \mathcal{M}_{g,1} \times \mathcal{M}_{g',1} \rightarrow \mathcal{M}_{g+g',1}.$$

*Proof.* Proof is given by refining that of Proposition 3.3. Consider the following commutative diagram

$$\begin{array}{ccccc}
& & \mathcal{M}_{g,1} \times \mathcal{M}_{g',1} & \xrightarrow{j \times j} & \text{Aut } F_{2g} \times \text{Aut } F_{2g'} \\
& & \mu_0 \downarrow & & \mu \downarrow \\
\mathcal{M}_{g,1} \times \mathcal{M}_{g',1} & \xrightarrow{\mu_0} & \mathcal{M}_{g+g'} & \xrightarrow{j} & \text{Aut } F_{2g+2g'} \\
p_0 \times p_0 \downarrow & & p_0 \downarrow & & p \downarrow \\
\text{Sp}(2g, \mathbb{Z}) \times \text{Sp}(2g', \mathbb{Z}) & \xrightarrow{\bar{\mu}_0} & \text{Sp}(2g+2g', \mathbb{Z}) & \xrightarrow{\bar{j}} & \text{GL}(2g+2g', \mathbb{Z}) \\
\bar{j} \times \bar{j} \downarrow & & \bar{j} \downarrow & & \\
\text{GL}(2g, \mathbb{Z}) \times \text{GL}(2g', \mathbb{Z}) & \xrightarrow{\bar{\mu}} & \text{GL}(2g+2g', \mathbb{Z}) & & 
\end{array}$$

together with the commutative diagram (1) where we replace  $n$  and  $n'$  with  $2g$  and  $2g'$  respectively. By the definition

$$\hat{\beta}_{2k+1}^0 = [j^* z_{4k} - p_0^* y_{4k}]$$

so that we have to compute

$$(2) \quad \mu_0^* \hat{\beta}_{2k+1}^0 = \mu_0^* [j^* z_{4k} - p_0^* y_{4k}].$$

First, we consider the first term in the above expression. We have proved in Proposition 3.3 that there exist cochains

$$\begin{aligned}
d &\in C^{4k-1}(\text{GL}(2g, \mathbb{Z}) \times \text{GL}(2g', \mathbb{Z}); \mathbb{R}), \\
d' &\in C^{4k-1}(\text{Aut } F_{2g} \times \text{Aut } F_{2g'}; \mathbb{R})
\end{aligned}$$

such that

$$\mu^* z_{4k} = z_{4k} \times 1 + 1 \times z_{4k} + (p \times p)^* d + \delta d'.$$

It follows that

$$\begin{aligned}
(3) \quad \mu_0^* j^* z_{4k} &= (j \times j)^* \mu^* z_{4k} \\
&= j^* z_{4k} \times 1 + 1 \times j^* z_{4k} + (j \times j)^* (p \times p)^* d + \delta (j \times j)^* d'.
\end{aligned}$$

Next we consider the second term of (2). By the definition we have  $\bar{j}^* \beta_{2k+1}^0 = \delta y_{4k}$ . Hence

$$(4) \quad \bar{\mu}_0^* \bar{j}^* b_{2k+1} = \bar{\mu}_0^* \delta y_{4k} = \delta \bar{\mu}_0^* y_{4k}.$$

On the other hand

$$\begin{aligned}
(5) \quad \bar{\mu}_0^* \bar{j}^* b_{2k+1} &= (\bar{j} \times \bar{j})^* \bar{\mu}^* b_{2k+1} \\
&= (\bar{j} \times \bar{j})^* (b_{2k+1} \times 1 + 1 \times b_{2k+1} + \delta d) \\
&= \delta y_{4k} \times 1 + 1 \times \delta y_{4k} + \delta (\bar{j} \times \bar{j})^* d.
\end{aligned}$$

From (4) and (5), we obtain

$$\delta (\bar{\mu}_0^* y_{4k} - y_{4k} \times 1 - 1 \times y_{4k} - (\bar{j} \times \bar{j})^* d) = 0.$$

Therefore, if we set

$$d'' = \bar{\mu}_0^* y_{4k} - y_{4k} \times 1 - 1 \times y_{4k} - (\bar{j} \times \bar{j})^* d,$$

then  $d''$  is a cocycle of the group  $\mathrm{Sp}(2g, \mathbb{Z}) \times \mathrm{Sp}(2g', \mathbb{Z})$  so that we can consider its cohomology class

$$[d''] \in H^{4k}(\mathrm{Sp}(2g, \mathbb{Z}) \times \mathrm{Sp}(2g', \mathbb{Z}); \mathbb{R})$$

and

$$\bar{\mu}_0^* y_{4k} = y_{4k} \times 1 + 1 \times y_{4k} + (\bar{j} \times \bar{j})^* d + d''.$$

It follows that

$$(6) \quad \begin{aligned} \mu_0^* p_0^* y_{4k} &= (p_0 \times p_0)^* \bar{\mu}_0^* y_{4k} \\ &= (p_0 \times p_0)^* (y_{4k} \times 1 + 1 \times y_{4k} + (\bar{j} \times \bar{j})^* d + d''). \end{aligned}$$

By combining (3) and (6), we obtain

$$(7) \quad \begin{aligned} \mu_0^* j^* z_{4k} - \mu_0^* p_0^* y_{4k} &= j^* z_{4k} \times 1 + 1 \times j^* z_{4k} + (j \times j)^* (p \times p)^* d + \delta(j \times j)^* d' \\ &\quad - (p_0 \times p_0)^* (y_{4k} \times 1 + 1 \times y_{4k} + (\bar{j} \times \bar{j})^* d + d'') \\ &= (j^* z_{4k} - p_0^* y_{4k}) \times 1 + 1 \times (j^* z_{4k} - p_0^* y_{4k}) \\ &\quad + \delta(j \times j)^* d' - (p_0 \times p_0)^* d''. \end{aligned}$$

Here we have used the equality

$$(j \times j)^* (p \times p)^* d = (p_0 \times p_0)^* (\bar{j} \times \bar{j})^* d$$

which follows from the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{M}_{g,1} \times \mathcal{M}_{g',1} & \xrightarrow{j \times j} & \mathrm{Aut} F_{2g} \times \mathrm{Aut} F_{2g'} \\ p_0 \times p_0 \downarrow & & p \times p \downarrow \\ \mathrm{Sp}(2g, \mathbb{Z}) \times \mathrm{Sp}(2g', \mathbb{Z}) & \xrightarrow{\bar{j} \times \bar{j}} & \mathrm{GL}(2g, \mathbb{Z}) \times \mathrm{GL}(2g', \mathbb{Z}). \end{array}$$

By combining (2) with (7) above, we now conclude that

$$(8) \quad \begin{aligned} \mu_0^* \hat{\beta}_{2k+1}^0 &= \mu_0^* [j^* z_{4k} - p_0^* y_{4k}] \\ &= [j^* z_{4k} - p_0^* y_{4k}] \times 1 + 1 \times [j^* z_{4k} - p_0^* y_{4k}] - (p_0 \times p_0)^* [d''] \\ &= \hat{\beta}_{2k+1}^0 \otimes 1 + 1 \otimes \hat{\beta}_{2k+1}^0 - (p_0 \times p_0)^* [d'']. \end{aligned}$$

By the definition of the class  $\hat{\beta}_{2k+1}^0$  again, it contains no monomials of MMM classes of *odd* indices, namely those of the form  $e_{2i_1-1}^{j_1} \cdots e_{2i_s-1}^{j_s}$ . Also any MMM class  $e_i$  is primitive (see [41] [42]) so that

$$\mu_0^* e_i = e_i \otimes 1 + 1 \otimes e_i \in H^{4i}(\mathcal{M}_{g,1} \times \mathcal{M}_{g',1}; \mathbb{Q}).$$

It follows that the class  $\mu_0^* \hat{\beta}_{2k+1}^0$  does not contain any term of the following form

$$(9) \quad e_{2i_1-1}^{j_1} \cdots e_{2i_s-1}^{j_s} \otimes e_{2i'_1-1}^{j'_1} \cdots e_{2i'_t-1}^{j'_t}.$$

Now as was recalled in §2, Borel [3] proved that

$$H^*(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{Q}) \cong \mathbb{Q}[c_1, c_3, \dots]$$

in a certain stable range. It follows that

$$H^*(\mathrm{Sp}(2g, \mathbb{Z}) \times \mathrm{Sp}(2g', \mathbb{Z}); \mathbb{Q}) \cong \mathbb{Q}[c_1, c_3, \dots] \otimes \mathbb{Q}[c_1, c_3, \dots]$$

again in a certain stable range. The cohomology class  $[d''']$  appearing in (8) belongs to this group but with the *real* coefficients.

On the other hand, as was already mentioned above, the homomorphism

$$p_0^* : H^*(\mathrm{Sp}(2g, \mathbb{Z}); \mathbb{Q}) \rightarrow H^*(\mathcal{M}_{g,1}; \mathbb{Q})$$

is known to be injective, in a certain stable range, and its image is precisely the subalgebra of  $\mathbb{Q}[e_1, e_2, \dots]$  generated by  $e_{\text{odd}}$  classes. Hence the class  $(p_0 \times p_0)^*[d''']$  is a linear combination of terms of the form described in (9).

By the above argument, we finally conclude that  $[d'''] = 0$  and hence

$$\mu_0^* \hat{\beta}_{2k+1}^0 = \hat{\beta}_{2k+1}^0 \otimes 1 + 1 \otimes \hat{\beta}_{2k+1}^0$$

as required. This completes the proof.  $\square$

The following is the main theorem of this section.

**Theorem 4.7.** *The secondary class  $\hat{\beta}_{2k+1}^0 \in H^{4k}(\mathcal{M}_{g,1}; \mathbb{R})$  is a non-zero multiple of the  $2k$ -th MMM class  $e_{2k}$ . More precisely*

$$\hat{\beta}_{2k+1}^0 = (-1)^k \zeta(2k+1) \frac{e_{2k}}{2(2k)!}.$$

*Proof.* By the previous proposition,  $\hat{\beta}_{2k+1}^0$  is a primitive cohomology class contained in  $H^{4k}(\mathcal{M}_{g,1}; \mathbb{R})$ . On the other hand, by the result of Madsen and Weiss (Theorem 2.4), the stable rational cohomology group of the mapping class group  $\mathcal{M}_{g,1}$  is isomorphic to the polynomial algebra generated by the MMM classes  $e_i$  all of which are primitive as mentined above. Hence we conclude that  $\hat{\beta}_{2k+1}^0$  is a multiple of  $e_{2k}$ .

The final part of the proof, namely the non-triviality of our class  $\hat{\beta}_{2k+1}^0$ , relies crucially on two results of Igusa given in [28]. One is his construction of a map

$$(10) \quad \mathrm{BOut}^{sp} F_{2g} \rightarrow |\mathrm{Wh}^h(\mathbb{Z}, 1)|$$

which induces the higher torsion classes of the group  $\mathrm{Out}^{sp} F_{2g}$ . The other is his determination (with Hain and Penner) of the higher torsion classes of the mapping class group (Theorem 8.5.10 of [28]). Here Igusa mentions that, although his axiomatic higher torsion ([30]) is only defined on the Torelli group  $\mathcal{I}_{g,*}$  and not on the mapping class group (surface bundles are not unipotent bundles in general), a higher FR torsion is defined on  $\mathcal{M}_{g,*}$  as well by using, what he calls, a framed function on even fiberwise suspension of the total spaces of surface bundles with sections. We refer to Igusa's books [28][29] for details.

Now we would like to consider the above mapping (10) in our context by examining the following homotopy commutative diagram.

$$\begin{array}{ccccccc}
& & & & \Omega \mathrm{BGL}(\infty, \mathbb{Z})^+ & & \\
& & & & \downarrow \bar{i}_0 & & \\
\mathrm{B}\mathcal{I}_{g,*} & \xrightarrow{\mathrm{B}j_0} & \mathrm{B}\mathrm{Out}_{2g} & \xlongequal{\quad} & \mathrm{B}\mathrm{Out}_{2g} & \xrightarrow{f_0} & |\mathrm{Wh}^h(\mathbb{Z}, 1)| \\
\mathrm{B}i_0 \downarrow & & \mathrm{B}i_0^{sp} \downarrow & & \mathrm{B}i \downarrow & & \downarrow \bar{i} \\
\mathrm{B}\mathcal{M}_{g,*} & \xrightarrow{\mathrm{B}j} & \mathrm{B}\mathrm{Out}^{sp} F_{2g} & \xrightarrow{\mathrm{B}j^{sp}} & \mathrm{B}\mathrm{Out} F_{2g} & \xrightarrow{f} & \mathbb{Z} \times \mathrm{B}\mathrm{Out}_{\infty}^+ \stackrel{h.e.}{\cong} QS^0 \\
\mathrm{B}p_0 \downarrow & & \mathrm{B}p_0^{sp} \downarrow & & \mathrm{B}p \downarrow & & \downarrow \bar{p} \\
\mathrm{BSp}(2g, \mathbb{Z}) & \xlongequal{\quad} & \mathrm{BSp}(2g, \mathbb{Z}) & \xrightarrow{\mathrm{B}\bar{j}} & \mathrm{BGL}(2g, \mathbb{Z}) & \xrightarrow{\bar{f}} & \mathbb{Z} \times \mathrm{BGL}(\infty, \mathbb{Z})^+.
\end{array}$$

By the result of Borel, the composed mapping  $\bar{f} \circ \mathrm{B}\bar{j}$  is rationally homotopic to the constant map when we let  $g$  go to the infinity. We choose a homotopy

$$H(\cdot, t) : \mathrm{BSp}(2\infty, \mathbb{Z}) \times I \rightarrow \mathbb{Z} \times \mathrm{BGL}(\infty, \mathbb{Z})^+$$

such that  $H(\cdot, 0) = \bar{f} \circ \mathrm{B}\bar{j}$  and  $H(\cdot, 1)$  is the constant map and let

$$\tilde{H}(\cdot, t) : \mathrm{B}\mathrm{Out}^{sp} F_{2\infty} \times I \rightarrow \mathbb{Z} \times \mathrm{B}\mathrm{Out}_{\infty}^+ \stackrel{h.e.}{\cong} QS^0$$

be the mapping which covers  $H$  and  $\tilde{H}(\cdot, 0) = f \circ \mathrm{B}j^{sp}$ . Then the mapping  $\tilde{H}(\cdot, 1)$  gives the desired mapping (10) (but defined only over the rationals). In this situation, we can refine the argument of the proof of Theorem 3.5. More precisely, taking the homotopies  $H, \tilde{H}$  into account we adjust the cochain  $j^* z_{4k}$  by subtracting  $p_0^* y_{4k}$  to make a cocycle. We then conclude that our construction of the secondary class  $\hat{\beta}_{2k}^0$  “realizes” the higher torsion classes induced by (10) in the framework of group cocycles.

Then the non-triviality as well as the precise constant follows from the determination of the higher torsion classes mentioned above.  $\square$

**Remark 4.8.** We have given the proof of Theorem 4.7 above within the framework of the theory of cohomology of groups as much as possible, more precisely except for the non-triviality (which is the most important property of course). It should be desirable to have a proof of the non-triviality purely in the context of the present paper.

**Remark 4.9.** We call our classes *secondary* characteristic classes associated with the vanishing of the Borel classes. However, the Borel classes are already secondary classes associated with the vanishing of the Chern classes on certain flat bundles. Therefore our classes are, so to speak, *secondary secondary* classes. Our result shows that these classes go back to the primary classes of the mapping class group, namely the MMM classes (of even indices).

## 5. CONJECTURAL GEOMETRIC MEANING OF THE MORITA CLASSES

In this section, we propose a conjectural meaning of the Morita classes

$$\mu_k \in H_{4k}(\mathrm{Out} F_{2k+2}; \mathbb{Q}) \quad (k = 1, 2, \dots)$$



which were introduced in [45] by making essential use of the foundational works of Culler and Vogtmann [13] and Kontsevich [35][36]. It was conjectured there that all the classes are non-trivial. At present, only the first three classes have been proved to be non-trivial ([45][11][21]).

We expect that the Morita classes will be detected by certain secondary classes associated with the difference between two reasons for the vanishing of Borel regulator classes. The definition of our secondary classes is given in a similar way as the case of the mapping class group treated in the previous section (although our actual development was done in the reverse order). More precisely, in that case we made use of the vanishing of  $\beta_{2k+1}$  on  $\text{Out } F_n$  as well as on  $\text{Sp}(2g, \mathbb{Z})$ , while in the present case we replace  $\text{Sp}(2g, \mathbb{Z})$  with  $\text{GL}(2k+2, \mathbb{Z})$  where  $2k+2$  is our conjectural optimal rank where  $\beta_{2k+1}$  vanishes.

Let us first recall a paper [37] by Lee where he mentioned that  $\beta_{2k+1}$  does not vanish in  $H^{4k+1}(\text{GL}(n, \mathbb{Z}); \mathbb{R})$  for all  $n \geq 2k+3$ . On the other hand, we have the following vanishing result.

**Theorem 5.1** (Bismut-Lott [2], Lee [37], Franke [18]). *For any integer  $k = 1, 2, \dots$ , the Borel regulator class  $\beta_{2k+1}$  vanishes in  $H^{4k+1}(\text{GL}(2k+1, \mathbb{Z}); \mathbb{R})$ .*

Thus the remaining open problem for the (non-)triviality of  $\beta_{2k+1}$  is as follows.

**Problem 5.2.** For each integer  $k \geq 1$ , determine whether the Borel class  $\beta_{2k+1}$  vanish in  $H^{4k+1}(\text{GL}(2k+2, \mathbb{Z}); \mathbb{R})$  or not.

The first two cases of the above problem have been solved. Lee and Szczarba [38] proved that  $H^5(\text{GL}(4, \mathbb{Z}); \mathbb{Q}) = 0$  so that  $\beta_3 = 0 \in H^5(\text{GL}(4, \mathbb{Z}); \mathbb{R})$ . Also Elbaz-Vincent-Gangl-Soulé [17] proved that  $H^9(\text{GL}(6, \mathbb{Z}); \mathbb{Q}) = 0$  so that  $\beta_5 = 0 \in H^9(\text{GL}(6, \mathbb{Z}); \mathbb{R})$ .

Based on these results, we would like to propose the following.

**Conjecture 5.3.** For any integer  $k = 1, 2, \dots$ , the Borel regulator class  $\beta_{2k+1}$  vanishes in  $H^{4k+1}(\text{GL}(2k+2, \mathbb{Z}); \mathbb{R})$ .

Assuming this conjecture, we define secondary cohomology classes

$$\overset{\circ}{\beta}_{2k+1} \in H^{4k}(\text{Aut } F_{2k+2}; \mathbb{R})$$

as follows. Let  $b_{2k+1} \in Z^{4k+1}(\text{GL}(n, \mathbb{Z}); \mathbb{R})$  be a cocycle representing  $\beta_{2k+1}$  as before and consider the following commutative diagram.

$$\begin{array}{ccc} \text{Aut } F_{2k+2} & \xrightarrow{j} & \text{Aut } F_n \\ p_0 \downarrow & & \downarrow p \\ \text{GL}(2k+2, \mathbb{Z}) & \xrightarrow{\bar{j}} & \text{GL}(n, \mathbb{Z}). \end{array}$$

Then, again as before, there exists a cochain  $z_{4k} \in C^{4k}(\text{Aut } F_n; \mathbb{R})$  such that

$$p^* b_{2k+1} = \delta z_{4k}.$$

Next  $\bar{j}^* \beta_{2k+1} = 0 \in H^{4k+1}(\mathrm{GL}(2k+2, \mathbb{Z}); \mathbb{R})$  by the assumption. Hence there exists a cochain  $x_{4k} \in C^{4k}(\mathrm{GL}(2k+2, \mathbb{Z}); \mathbb{R})$  such that

$$\bar{j}^* b_{2k+1} = \delta x_{4k}.$$

Then we have

$$\delta(j^* z_{4k} - p_0^* x_{4k}) = j^* p^* b_{2k+1} - p_0^* \bar{j}^* b_{2k+1} = 0$$

so that  $(j^* z_{4k} - p_0^* x_{4k})$  is a  $4k$ -cocycle of the group  $\mathrm{Aut} F_{2k+2}$ .

**Definition 5.4.** We define

$$\overset{\circ}{\beta}_{2k+1} = [j^* z_{4k} - p_0^* x_{4k}] \in H^{4k}(\mathrm{Aut} F_{2k+2}; \mathbb{R}).$$

**Proposition 5.5.** *The cohomology class  $\overset{\circ}{\beta}_{2k+1}$  is well-defined independent of the choices of  $b_{2k+1}, z_{4k}$  and  $x_{4k}$  modulo the indeterminacy*

$$\mathrm{Im} \left( H^{4k}(\mathrm{GL}(2k+2, \mathbb{Z}); \mathbb{R}) \rightarrow H^{4k}(\mathrm{Aut} F_{2k+2}; \mathbb{R}) \right).$$

*Proof.* First observe that we can add any  $4k$ -cocycle of  $\mathrm{GL}(2k+2, \mathbb{Z})$  to a given  $x_{4k}$  so that the cohomology class  $\overset{\circ}{\beta}_{2k+1}$  can vary freely within the indeterminacy. Now let  $b'_{2k+1}, z'_{4k}, x'_{4k}$  be another set of choices so that

$$p^* b'_{2k+1} = \delta z'_{4k}, \quad \bar{j}^* b'_{2k+1} = \delta x'_{4k}.$$

Then, as before, there exist elements  $u \in C^{4k}(\mathrm{GL}(n, \mathbb{Z}); \mathbb{R})$  and  $v \in C^{4k-1}(\mathrm{Out} F_n; \mathbb{R})$  such that

$$b'_{2k+1} = b_{2k+1} + \delta u \quad \text{and} \quad z'_{4k} = z_{4k} + p^* u + \delta v.$$

It follows that

$$\delta x'_{4k} = \bar{j}^* b'_{2k+1} = \bar{j}^* (b_{2k+1} + \delta u) = \delta x_{4k} + \bar{j}^* \delta u$$

and so  $(x'_{4k} - x_{4k} - u)$  is a cocycle of  $\mathrm{GL}(2k+2, \mathbb{Z})$  and we have

$$[x'_{4k} - x_{4k} - \bar{j}^* u] \in H^{4k}(\mathrm{GL}(2k+2, \mathbb{Z}); \mathbb{R}).$$

On the other hand

$$\begin{aligned} (j^* z'_{4k} - p_0^* x'_{4k}) - (j^* z_{4k} - p_0^* x_{4k}) &= j^* (p^* u + \delta v) - p_0^* (x'_{4k} - x_{4k}) \\ &= -p_0^* (x'_{4k} - x_{4k} - \bar{j}^* u) + \delta j^* v. \end{aligned}$$

Hence

$$[j^* z'_{4k} - p_0^* x'_{4k}] = [j^* z_{4k} - p_0^* x_{4k}] - p_0^* [x'_{4k} - x_{4k} - \bar{j}^* u] \in H^{4k}(\mathrm{Aut} F_{2k+2}; \mathbb{R}).$$

Therefore

$$[j^* z'_{4k} - p_0^* x'_{4k}] = [j^* z_{4k} - p_0^* x_{4k}] \quad \text{mod indeterminacy}$$

as required.  $\square$

There is a close relation between the secondary classes  $\overset{\circ}{\beta}_{2k+1}$  and  $T\beta_{2k+1}$  as below, which shows that the class  $\overset{\circ}{\beta}_{2k+1}$  can be interpreted as an “extension” of  $T\beta_{2k+1}$  to the whole group  $\mathrm{Aut} F_{2k+2}$  at the “critical rank”  $n = 2k + 2$ .

**Proposition 5.6.** *Let  $i : \text{IA}_{2k+2} \subset \text{Aut } F_{2k+2}$  be the inclusion. Then for any choice of  $\overset{\circ}{\beta}_{2k+1}$  within its indeterminacy, we have the identity*

$$i^* \overset{\circ}{\beta}_{2k+1} = T\beta_{2k+1} \in H^{4k}(\text{IA}_{2k+2}; \mathbb{R})^{\text{GL}(2k+2, \mathbb{Z})}.$$

*Proof.* This follows easily from the definitions of the two classes, because  $i^* p_0^* x_{4k} = 0$  for any  $x_{4k}$ .  $\square$

To state our conjecture, recall that Conant and Vogtmann [11] proved that the class  $\mu_k$  has a natural lift in  $H_{4k}(\text{Aut } F_{2k+2}; \mathbb{Q})$  under the projection  $\text{Aut } F_{2k+2} \rightarrow \text{Out } F_{2k+2}$  which we denote by the same letter. We refer to [33] for the relation between the rational homology groups of  $\text{Aut } F_n$  and  $\text{Out } F_n$  in general.

**Conjecture 5.7.** For a suitable choice of  $\overset{\circ}{\beta}_{2k+1}$  within the indeterminacy, we have

$$\langle \overset{\circ}{\beta}_{2k+1}, \mu_k \rangle \neq 0.$$

**Remark 5.8.** If the projected image  $(p_0)_*(\mu_k) \neq 0 \in H_{4k}(\text{GL}(2k+2, \mathbb{Z}); \mathbb{Q})$ , then the above conjecture holds “trivially” because we can add to  $\overset{\circ}{\beta}_{2k+1}$  the pull back under  $p_0^*$  of an element in  $H^{4k}(\text{GL}(2k+2, \mathbb{Z}); \mathbb{Q})$  which detects  $(p_0)_*(\mu_k)$ . On the other hand, if  $(p_0)_*(\mu_k) = 0$ , then it is easy to see that the value  $\langle \overset{\circ}{\beta}_{2k+1}, \mu_k \rangle$  does not depend on the choice of  $\overset{\circ}{\beta}_{2k+1}$  within the indeterminacy.

We have also the following “dual version” of the above argument. It applies to the case  $(p_0)_*(\mu_k) = 0$  and is based on the following two results. One is due to Conant and Vogtmann [12] who proved that  $\mu_k \in H_{4k}(\text{Aut } F_{2k+2}; \mathbb{Q})$  vanishes in  $H_{4k}(\text{Aut } F_{2k+3}; \mathbb{Q})$  under the natural inclusion  $F_{2k+2} \subset F_{2k+3}$ . The other is one particular result, in a recent remarkable paper [10] by Conant, Hatcher, Kassabov and Vogtmann, that  $\mu_k$  is supported on a certain free abelian subgroup  $\mathbb{Z}^{4k} \subset \text{Out } F_{2k+2}$ . Here they also give a new proof of the above vanishing result under one stabilization.

Our strategy is as follows. By using the above mentioned result in [10], we can construct an explicit  $(2k+3)$ -chain  $u_f \in C_{2k+3}(\text{Out } F_{2k+3}; \mathbb{Q})$  which bounds the abelian cycle  $j(\mathbb{Z}^{4k}) \subset \text{Out } F_{2k+3}$ . On the other hand, by the assumption  $(p_0)_*(\mu_k) = 0$ , there exists a chain  $u_b \in C_{2k+3}(\text{GL}(2k+2, \mathbb{Z}); \mathbb{Q})$  which bounds the abelian cycle  $(p_0)_*(\mathbb{Z}^{4k}) \subset \text{GL}(2k+2, \mathbb{Z})$  (it can be checked that  $(p_0)_*$  is injective on  $\mathbb{Z}^{4k}$ ). Then consider the element

$$u = p_*(u_f) - \bar{j}_*(u_b) \in Z_{4k+1}(\text{GL}(2k+3, \mathbb{Z}); \mathbb{Q})$$

which is a cycle because

$$\partial u = \partial u_f - \partial u_b = p_* j_*(\mathbb{Z}^{4k}) - \bar{j}_*(p_0)_*(\mathbb{Z}^{4k}) = 0.$$

**Conjecture 5.9.** For a suitable choice of  $u$ , we have

$$\langle \beta_{2k+1}, [u] \rangle \neq 0.$$

**Proposition 5.10.** *If Conjecture 5.9 holds for  $k$ , then  $\mu_k \neq 0$ .*

*Proof.* It suffices to prove the following. If  $\mu_k = 0$ , then  $\langle \beta_{2k+1}, [u] \rangle = 0$  for any choice of the cycle  $u$ . By the assumption that  $\mu_k = 0$ , there exists a chain  $u_0 \in C_{4k+1}(\text{Out } F_{2k+2}; \mathbb{Q})$  which bounds  $\mathbb{Z}^{4k} \subset \text{Out } F_{2k+2}$ . Then we can set  $u_f = j_*(u_0)$  and  $u_b = (p_0)_*(u_0)$  in the above construction of the cycle  $u$ , which implies  $u = 0$ . On the other hand, in this case the indeterminacy of  $u$  comes from adding certain  $(4k+1)$ -cycles of the two groups  $\text{Out } F_{2k+3}$  and  $\text{GL}(2k+2, \mathbb{Z})$  on which the Borel class  $\beta_{2k+1}$  takes the value 0, the former group by the vanishing theorem of Igusa and Galatius and the latter group by the assumption that Conjecture 5.3 holds. This completes the proof.  $\square$

**Remark 5.11.** We are planning to prove Conjecture 5.9 for the case  $k = 1$  by using Hamida's cocycle for  $\beta_{2k+1}$  given in [22] and a computer computation along the line of [6].

## 6. PROSPECTS AND FINAL REMARKS

Here we discuss a relationship between our secondary classes  $T\beta_{2k+1}, \hat{\beta}_{2k+1}$  and an important open question about the cohomology of the Torelli group  $\mathcal{I}_g$  whether the MMM classes of *even* indices are non-trivial in  $H^{4*}(\mathcal{I}_g; \mathbb{Q})$  or not. We mention that even the non-triviality of  $e^2 \in H^4(\mathcal{I}_{g,*}; \mathbb{Q})$  is not known where  $\mathcal{I}_{g,*}$  denotes the Torelli group of a genus  $g$  surface with base point and  $e \in H^2(\mathcal{I}_{g,*}; \mathbb{Z})$  denotes the Euler class of the tangent bundle along the fibers of surface bundles. See Sakasai [52] for an attempt to attack this problem and Salter [54] for a more recent work.

**Conjecture 6.1** (Church and Farb [7], Conjecture 6.5). The  $\text{GL}(n, \mathbb{Z})$ -invariant part of the stable rational cohomology of  $\text{IA}_n$  vanishes.

We would like to point out that an affirmative solution to the above conjecture implies that all the MMM classes of *even* indices are trivial on the Torelli group. Indeed, our secondary classes  $T\beta_{2k+1}^0 \in H^{4k}(\text{IA}_n; \mathbb{R})^{\text{GL}(n, \mathbb{Z})}$  are stable classes and  $\text{GL}(n, \mathbb{Z})$ -invariant (see Proposition 3.1 and Definition 3.2). Therefore if we assume the above conjecture, then we have  $T\beta_{2k+1}^0 = 0$ . On the other hand, Theorem 3.5 shows that  $T\beta_{2k+1}^0$  is equal to Igusa's higher torsion  $\tau_{2k}(\text{IA}_n)$ . We can then use Igusa's result in [28] that the restriction of  $\tau_{2k}(\text{IA}_n)$  to the Torelli group  $\mathcal{I}_{g,1}$  is a non-zero multiple of  $e_{2k}$  to conclude that this class vanishes.

If we consider the spectral sequence for the rational cohomology group of the extension

$$1 \rightarrow \text{IA}_n \rightarrow \text{Aut } F_n \rightarrow \text{GL}(n, \mathbb{Z}) \rightarrow 1,$$

the  $E_2$  term is given by

$$E_2^{p,q} = H^p(\text{GL}(n, \mathbb{Z}); H^q(\text{IA}_n; \mathbb{Q}))$$

and the Borel classes are described as

$$\beta_{2k+1} \neq 0 \in E_2^{4k+1,0}.$$

Because of the vanishing theorem of Galatius, these classes do not survive in the  $E_\infty$  term. Hence among the following groups

$$(11) \quad H^{4k-i}(\mathrm{GL}(n, \mathbb{Z}); H^i(\mathrm{IA}_n; \mathbb{Q})) \quad (i = 1, \dots, 4k),$$

there should exist at least one non-trivial group that kills  $\beta_{2k+1}$ . The secondary class

$$T\beta_{2k+1}^0 = \tau_{2k}(\mathrm{IA}_n) \in H^{4k}(\mathrm{IA}_n; \mathbb{R})^{\mathrm{GL}(n, \mathbb{Z})} \cong H^0(\mathrm{GL}(n, \mathbb{Z}); H^{4k}(\mathrm{IA}_n; \mathbb{R}))$$

corresponds to the last case of  $i = 4k$ .

Conjecture 6.1, applied to  $\beta_{2k+1}^0$ , is equivalent to saying that it is not killed in the last step, namely  $T\beta_{2k+1}^0 = 0$  so that it must be killed somewhere in the range  $1 \leq i \leq 4k-1$ . We would like to ask the validity of, so to speak, the most optimistic possibility.

**Question 6.2.** Are the secondary classes  $T\beta_{2k+1}^0 \in H^{4k}(\mathrm{IA}_n; \mathbb{R})^{\mathrm{GL}(n, \mathbb{Z})}$  non-trivial in a certain stable range? If not, which group in (11) ( $1 \leq i \leq 4k-1$ ) kills  $\beta_{2k+1}$ ?

**Remark 6.3.** There have been obtained several important results concerning the homology of the group  $\mathrm{IA}_n$  including [33][51][1][9][14]. However, no non-trivial elements in  $H^*(\mathrm{IA}_n; \mathbb{Q})^{\mathrm{GL}(n, \mathbb{Z})}$  were found. The problem of determining whether this group is non-trivial or not remains a mystery.

**Remark 6.4.** We think that the non-triviality of the Morita classes is a very important problem not only in the theory of cohomology of automorphism groups of free groups but also in low dimensional geometric topology. This is because, we expect that the “group version” defined in [47], of these classes (the Lie algebra version) would detect the difference between the smooth and topological categories in four-dimensional topology (see [46][49]), although it will need many more years for this expectation to be clarified.

**Remark 6.5.** In this paper, we defined our secondary classes by making use of two different ways of vanishing of the Borel regulator classes. It would be meaningful to recall here the following work. Namely, by making use of two different cocycles representing the first MMM class, the first named author defined in [43] a secondary characteristic class which is an  $\mathcal{M}_g$ -invariant homomorphism

$$d : \mathcal{K}_g \rightarrow \mathbb{Q}$$

where  $\mathcal{K}_g \subset \mathcal{M}_g$  denotes the kernel of the Johnson homomorphism [31].

It was proved that this is a manifestation of the Casson invariant in the structure of the mapping class group.

**Remark 6.6.** More generally, by comparing two cocycles for the MMM classes  $e_{2k-1}$  of odd indices, one from  $\mathrm{Sp}(2g, \mathbb{Z})$  and the other given in [44][34], certain secondary classes

$$d_k \in H^{4k-3}(\mathcal{K}_g, \mathbb{Q})^{\mathcal{M}_g} \quad (k = 1, 2, \dots)$$

were defined in [45][46], where  $d_1$  is the same as  $d$  in the previous remark.

In this paper, we have constructed a new cocycle for the even class  $e_{2k}$  which is essentially different from the one given in [44][34]. It would be interesting to study

whether the difference between these two cocycles will give rise to certain secondary invariants for a suitable subgroup of the mapping class group.

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